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Phase-space metric for non-Hamiltonian systems

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Abstract
We consider an invariant skew-symmetric phase-space metric for non-Hamiltonian systems. We say that the metric is an invariant if the metric tensor field is an integral of motion. We derive the time-dependent skew-symmetric phase-space metric that satisfies the Jacobi identity. The example of non-Hamiltonian systems with linear friction term is considered.

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1. Introduction

The dynamics of Hamiltonian systems is characterized by conservation of phase-space volume under time evolution. This conservation of the phase volume is a cornerstone of conventional statistical mechanics of Hamiltonian systems. At a mathematical level, conservation of phase-space volume is considered as a consequence of the existence of an invariant symplectic form (skew-symmetric phase-space metric) in the phase-space of Hamiltonian systems [1–3].

The classical statistical mechanics of non-Hamiltonian systems is of strong theoretical interest [4–17]. Non-Hamiltonian systems have been used in molecular dynamics simulation to achieve the calculation of statistical averages in various ensemble [5, 6, 12, 18], and in the treatment of nonequilibrium steady states [16, 17, 19, 20]. Non-Hamiltonian systems are characterized by nonzero phase-space compressibility, and the usual phase-space volume is no longer necessarily conserved.

Tuckerman *et al* have argued [5, 6] that there is a measure conservation law that involves a nontrivial phase-space metric. This suggests that phase-space should be carefully treated using the general rules of the geometry of manifolds [1, 2]. Tuckerman *et al* have applied the concepts of Riemannian geometry to the classical statistical mechanics of non-Hamiltonian systems [4–6]. Tuckerman *et al* have argued that, through introduction of metric determinant factors $\sqrt{g(\mathbf{x},t)}$, it is possible to define an invariant phase-space measure for non-Hamiltonian systems. In their approach the metric determinant factor $\sqrt{g(\mathbf{x},t)}$, where $g(\mathbf{x},t)$ is the determinant of the metric tensor, is defined by the compressibility of non-Hamiltonian systems.

However Tuckerman *et al* consider only the determinant $g(\mathbf{x}, t)$ of the metric. The phase-space metric is not considered in [4–6]. Note that Tuckerman *et al* suppose that the metric determinant factor is connected with symmetric phase-space metric. It can be proved that the proposal to use an invariant time-dependent metric determinant factor in the volume element corresponds precisely to finding a skew-symmetric phase-space metric (symplectic form) that is an integral of motion. Therefore we must consider the skew-symmetric phase-space metric.

Sergi [10, 11] has considered an antisymmetric phase-space tensor field, whose elements are general function of phase-space coordinates. In [10, 11], the generalization of Poisson brackets for the non-Hamiltonian systems was suggested. However the Jacobi identity is not satisfied by the generalized brackets and skew-symmetric phase-space metric. As a result the algebra of phase-space functions is not time translation invariant. The generalized brackets do not define a Lie algebra in phase-space. Note that the generalized brackets of two constants of motion is no longer a constant of motion.

In the present paper we consider an invariant skew-symmetric (antisymmetric) phase-space metric for non-Hamiltonian systems. We say that the metric is an invariant, if the metric tensor field is an integral of motion. We define the phase-space metric such that the Jacobi identity is satisfied. The suggested skew-symmetric phase-space metric allows us to introduce the generalization of the Poisson brackets for non-Hamiltonian systems such that the Jacobi identity is satisfied by the generalized Poisson brackets. As a result the algebra of phase-space functions is time translation invariant. The generalized Poisson brackets define a Lie algebra in phase-space. The suggested Poisson brackets of two constants of motion is a constant of motion.

In section 2, the definitions of the antisymmetric phase-space metric, and mathematical background and notations are considered. In section 3, we define the non-Hamiltonian systems, and consider the Helmholtz conditions. In section 4, we consider the time evolution of phase-space metric. We derive the phase-space metric that is an integral of motion. In section 5, the generalized Poisson brackets for non-Hamiltonian systems are defined. In section 6, the example of phase-space metric for non-Hamiltonian system with the linear friction term is considered. Finally, a short conclusion is given in section 7.

2. Phase-space metric

The $2n$-dimensional differentiable manifold is denoted by M. Coordinates are $\mathbf{x} = (x^1, \ldots, x^{2n})$. We assume the existence of a time-dependent metric tensor field $\omega_{kl}(\mathbf{x}, t)$ on the manifold M. We can define a differential 2-form

$$\omega = \omega_{kl}(\mathbf{x}, t)\, \mathrm{d}x^k \wedge \mathrm{d}x^l, \tag{1}$$

where $\omega_{kl} = \omega_{kl}(\mathbf{x}, t)$ is a skew-symmetric tensor $\omega_{kl} = -\omega_{lk}$, and the tensor elements $\omega_{kl}(\mathbf{x}, t)$ are explicit functions of time. Here and later we mean the sum on the repeated index k and l from 1 to $2n$.

We suppose that the differential 2-form ω is a closed nondegenerated form:

(1) If the metric determinant is not equal to zero

$$g(\mathbf{x}, t) = \det(\omega_{kl}(\mathbf{x}, t)) \neq 0 \tag{2}$$

for all points $\mathbf{x} \in M$, then the form ω is nondegenerated.

(2) If the Jacobi identity

$$\partial_k \omega_{lm} + \partial_l \omega_{mk} + \partial_m \omega_{kl} = 0, \qquad \partial_k = \partial/\partial x^k \tag{3}$$

for the metric $\omega_{kl} = \omega_{kl}(\mathbf{x}, t)$ is satisfied, then the differential 2-form ω is closed ($\mathrm{d}\omega = 0$).

Phase space is therefore assumed to be a symplectic manifold.

Definition 1. *A* symplectic manifold *is a differentiable manifold M with a closed nondegenerated differential 2-form ω.*

For symplectic manifold, we have the phase-space volume element

$$v = \frac{1}{n!}\omega^n = \frac{1}{n!}\omega \wedge \cdots \wedge \omega.$$

This differentiable $2n$-form can be represented by

$$v = \sqrt{g(\mathbf{x}, t)}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{2n},$$

where $g(\mathbf{x}, t)$ is defined by equation (2). The nondegenerated condition ($g(\mathbf{x}, t) \neq 0$) for the metric $\omega_{kl}(\mathbf{x}, t)$ is equivalent to the condition $\omega^n \neq 0$ or $v \neq 0$.

It is known [2] that there exist the local coordinates (q, p) such that

$$\omega = \delta_{ij}\, \mathrm{d}q^i \wedge \mathrm{d}p^j. \tag{4}$$

Here and later we mean the sum on the repeated index i and j from 1 to n.

3. Non-Hamiltonian system

The dynamics is described by a smooth vector field $\mathbf{X} = \mathbf{X}(\mathbf{x})$,

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{X} \tag{5}$$

with components X^k in basis $\partial_k = \partial/\partial x^k$. For simplicity, we consider the case where the vector field $\mathbf{X}$ is time independent. In local coordinates $\{x^k\}$, equation (5) has the form

$$\frac{\mathrm{d}x^k}{\mathrm{d}t} = X^k. \tag{6}$$

Consider now the definition of the Hamiltonian systems, which is used in [1].

Definition 2. *A classical system (5) on the symplectic manifold (M, ω) is called a* Hamiltonian system *if the differential 1-form $\omega(X)$ is a closed form*

$$\mathrm{d}\omega(X) = 0,$$

where $\omega(X) = \mathrm{i}_X\omega$ is the contraction (interior product) of the 2-form ω with vector $\mathbf{X}$, and d is the exterior derivative.

A classical system (5) on the symplectic manifold (M, ω) is called a non-Hamiltonian system *if the differential 1-form $\omega(X)$ is nonclosed* $\mathrm{d}\omega(X) \neq 0$.

Proposition 1. *The classical system (5) is a Hamiltonian system if the conditions*

$$J_{kl}(\omega, \mathbf{x}, t) \equiv \partial_k(\omega_{lm}X^m) - \partial_l(\omega_{km}X^m) = 0 \tag{7}$$

are satisfied.

Proof. In the local coordinates $\{x^k\}$, we have

$$\omega(X) = X_k\, \mathrm{d}x^k = \omega_{kl}X^l\, \mathrm{d}x^k,$$

where $X_k = \omega_{kl}X^l$. In this case, the exterior derivative of 1-form $\omega(X)$ is

$$\mathrm{d}\omega(X) = \mathrm{d}(X_k\, \mathrm{d}x^k) = \partial_l X_k\, \mathrm{d}x^l \wedge \mathrm{d}x^k.$$

Using $\mathrm{d}a \wedge \mathrm{d}b = -\mathrm{d}b \wedge \mathrm{d}a$, we get

$$\mathrm{d}\omega(X) = \tfrac{1}{2}(\partial_k X_l - \partial_l X_k)\,\mathrm{d}x^k \wedge \mathrm{d}x^l.$$

As a result we have the differential 2-form

$$\mathrm{d}\omega(X) = \tfrac{1}{2}(\partial_k(\omega_{lm}X^m) - \partial_l(\omega_{km}X^m))\,\mathrm{d}x^k \wedge \mathrm{d}x^l. \tag{8}$$

This differential 2-form is a symplectic form, which can be called 'non-Hamiltonian symplectic form'. If the Helmholtz conditions (7) are satisfied, then the differential 1-form $\omega(X)$ is closed ($\mathrm{d}\omega(X) = 0$), and the classical system (5) is a Hamiltonian system.

Let us consider the canonical coordinates $\mathbf{x} = (x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n}) = (q^1, \ldots, q^n, p^1, \ldots, p^n)$. Equation (6) can be written as

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = G^i(q, p), \qquad \frac{\mathrm{d}p^i}{\mathrm{d}t} = F^i(q, p). \tag{9}$$

□

Corollary. *If the right-hand sides of equations (9) satisfy the Helmholtz conditions [21, 22] for the phase-space with (4), which have the following form:*

$$\frac{\partial G^i}{\partial p^j} - \frac{\partial G^j}{\partial p^i} = 0, \tag{10}$$

$$\frac{\partial G^j}{\partial q^i} + \frac{\partial F^i}{\partial p^j} = 0, \tag{11}$$

$$\frac{\partial F^i}{\partial q^j} - \frac{\partial F^j}{\partial q^i} = 0, \tag{12}$$

then classical system (9) is a Hamiltonian system.

Proof. In the canonical coordinates (q, p), the vector field $\mathbf{X}$ has the components (G^i, F^i), which are used in equation (9). The 1-form $\omega(X)$ is defined by the following equation:

$$\omega(X) = \tfrac{1}{2}(G_i\,\mathrm{d}p^i - F_i\,\mathrm{d}q^i),$$

where $G_i = \delta_{ij}G^j$ and $F_i = \delta_{ij}F^j$. The exterior derivative for this form can now be written by the relation

$$\mathrm{d}\omega(X) = \tfrac{1}{2}(\mathrm{d}(G_i dp^i) - \mathrm{d}(F_i\,\mathrm{d}q^i)).$$

It now follows that

$$\mathrm{d}\omega(X) = \frac{1}{2}\left(\frac{\partial G_i}{\partial q^j}\mathrm{d}q^j \wedge \mathrm{d}p^i + \frac{\partial G_i}{\partial p^j}\mathrm{d}p^j \wedge \mathrm{d}p^i - \frac{\partial F_i}{\partial q^j}\mathrm{d}q^j \wedge \mathrm{d}q^i - \frac{\partial F_i}{\partial p^j}\mathrm{d}p^j \wedge \mathrm{d}q^i\right).$$

This equation can be rewritten in an equivalent form

$$\mathrm{d}\omega(X) = \frac{1}{2}\left(\frac{\partial G_j}{\partial q^i} + \frac{\partial F_i}{\partial p^j}\right)\mathrm{d}q^i \wedge \mathrm{d}p^j + \frac{1}{4}\left(\frac{\partial G_j}{\partial p^i} - \frac{\partial G_i}{\partial p^j}\right)\mathrm{d}p^i \wedge \mathrm{d}p^j + \frac{1}{4}\left(\frac{\partial F_i}{\partial q^j} - \frac{\partial F_j}{\partial q^i}\right)\mathrm{d}q^i \wedge \mathrm{d}q^j.$$

Here we use the skew symmetry of $\mathrm{d}q^i \wedge \mathrm{d}q^j$ and $\mathrm{d}p^i \wedge \mathrm{d}p^j$ with respect to index i and j. It is obvious that conditions (10)–(12) lead to the equation $\mathrm{d}\omega(X) = 0$. □

4. Time evolution of phase-space metric

Let us find a time-dependent symplectic 2-form ω that satisfies the equation $\mathrm{d}\omega/\mathrm{d}t = 0$.

It is known [2, 3] as the following proposition.

Proposition 2. *If the system* $\dot{\mathbf{x}} = \mathbf{X}$ *on the symplectic manifold* (M, ω) *with time-independent symplectic form* $(\partial\omega_{kl}/\partial t = 0)$ *is a Hamiltonian system, then differential 2-form* ω *is conserved, i.e.,* $\mathrm{d}\omega/\mathrm{d}t = 0$.

Proof. The proof of this theorem is considered in [2, 3]. □

Let us consider a generalization of this proposition.

Proposition 3. *If the time-dependent metric* $\omega_{kl} = \omega_{kl}(\mathbf{x}, t)$ *is a skew-symmetric metric* $(\omega_{kl} = -\omega_{lk})$ *that is satisfied by the Jacobi identity (3), and the system is defined by equation (6), then the total time derivative of the differential 2-form (1) is given by*

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \left(\frac{\partial\omega_{kl}}{\partial t} - \partial_k(\omega_{lm}X^m) + \partial_l(\omega_{km}X^m)\right) \mathrm{d}x^k \wedge \mathrm{d}x^l. \tag{13}$$

Proof. The time-derivative of the time-dependent symplectic form ω is given by

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\omega_{kl}(\mathbf{x}, t)\,\mathrm{d}x^k \wedge \mathrm{d}x^l) = \frac{\mathrm{d}\omega_{kl}}{\mathrm{d}t}\,\mathrm{d}x^k \wedge \mathrm{d}x^l + \omega_{kl}\,\mathrm{d}\left(\frac{\mathrm{d}x^k}{\mathrm{d}t}\right) \wedge \mathrm{d}x^l + \omega_{kl}\,\mathrm{d}x^k \wedge \mathrm{d}\left(\frac{\mathrm{d}x^l}{\mathrm{d}t}\right).$$

Then, using the equation

$$\frac{\mathrm{d}\omega_{kl}}{\mathrm{d}t} = \frac{\partial\omega_{kl}}{\partial t} + \frac{\partial\omega_{kl}}{\partial x^m}\frac{\mathrm{d}x^m}{\mathrm{d}t} = \frac{\partial\omega_{kl}}{\partial t} + X^m\partial_m\omega_{kl},$$

and equation (6), we find that

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \left(\frac{\partial\omega_{kl}}{\partial t} + X^m\partial_m\omega_{kl}\right) \mathrm{d}x^k \wedge \mathrm{d}x^l + \omega_{kl}\,\mathrm{d}X^k \wedge \mathrm{d}x^l + \omega_{kl}\,\mathrm{d}x^k \wedge \mathrm{d}X^l.$$

Using $\mathrm{d}X^k = \partial_m X^k\,\mathrm{d}x^m$, we have

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \left(\frac{\partial\omega_{kl}}{\partial t} + X^m\partial_m\omega_{kl}\right) \mathrm{d}x^k \wedge \mathrm{d}x^l + \omega_{kl}\partial_m X^k\,\mathrm{d}x^m \wedge \mathrm{d}x^l + \omega_{kl}\partial_m X^l\,\mathrm{d}x^k \wedge \mathrm{d}x^m.$$

This expression can be rewritten in an equivalent form

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \left(\frac{\partial\omega_{kl}}{\partial t} + X^m\partial_m\omega_{kl} + \omega_{ml}\partial_k X^m + \omega_{km}\partial_l X^m\right) \mathrm{d}x^k \wedge \mathrm{d}x^l.$$

Using the rule of term-by-term differentiation in the form

$$\omega_{ml}\partial_k X^m = \partial_k(\omega_{ml}X^m) - X^m\partial_k\omega_{ml}, \qquad \omega_{km}\partial_l X^m = \partial_l(\omega_{km}X^m) - X^m\partial_l\omega_{km},$$

we get the following equation:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \left(\frac{\partial\omega_{kl}}{\partial t} + X^m(\partial_m\omega_{kl} - \partial_k\omega_{ml} - \partial_l\omega_{km}) + \partial_k(\omega_{ml}X^m) + \partial_l(\omega_{km}X^m)\right) \mathrm{d}x^k \wedge \mathrm{d}x^l.$$

Using the Jacobi identity (3), and skew symmetry $\omega_{ml} = -\omega_{lm}$, $\omega_{km} = -\omega_{mk}$, we have

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \left(\frac{\partial\omega_{kl}}{\partial t} - \partial_k(\omega_{lm}X^m) + \partial_l(\omega_{km}X^m)\right) \mathrm{d}x^k \wedge \mathrm{d}x^l.$$

As a result, we obtain equation (13) for the total time derivative of symplectic form. □

The total time derivative of the symplectic form is defined by equation (13). If the total derivative is zero, than we have the integral of motion or invariant. It is easy to see that the differentiable 2-form ω is invariant if the phase-space metric $\omega_{kl}(\mathbf{x}, t)$ is satisfied by the equation

$$\frac{\partial \omega_{kl}}{\partial t} = \partial_k(\omega_{lm} X^m) - \partial_l(\omega_{km} X^m). \tag{14}$$

This equation can be rewritten in an equivalent form

$$\frac{\partial \omega_{kl}}{\partial t} = \hat{J}_{kl}^{ms} \omega_{ms},$$

where the operator $\hat{J}$ is defined by the equation

$$\hat{J}_{kl}^{ms} = \frac{1}{2}\left(\left(\delta_l^m \partial_k - \delta_k^m \partial_l\right) X^s - \left(\delta_l^s \partial_k - \delta_k^s \partial_l\right) X^m\right). \tag{15}$$

Proposition 4. *The differentiable 2-form ω is invariant (is an integral of motion for non-Hamiltonian system (6)) if the phase-space metric $\omega_{kl}(\mathbf{x}, t)$ is defined by the equation*

$$\omega_{kl}(\mathbf{x}, t) = (\exp(t\hat{J}))_{kl}^{ms} \omega_{ms}(\mathbf{x}, 0). \tag{16}$$

Here $\hat{J}$ is an operator that is defined by equation (15).

Proof. Let us consider the formal solution of equation (14) in the form

$$\omega_{kl}(\mathbf{x}, t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \omega_{kl}^{(n)}(\mathbf{x}),$$

where $\omega_{kl}^{(n)} = -\omega_{lk}^{(n)}$. In this case, the time-independent tensor fields $\omega_{kl}^{(n)}(\mathbf{x})$ are defined by the recursion relation

$$\omega_{kl}^{(n+1)} = \partial_k\left(\omega_{lm}^{(n)} X^m\right) - \partial_l\left(\omega_{km}^{(n)} X^m\right).$$

This equation can be rewritten in an equivalent form

$$\omega_{kl}^{(n+1)} = \left(\delta_l^m \partial_k - \delta_k^m \partial_l\right)\left(X^s \omega_{ms}^{(n)}\right).$$

Using the skew symmetry of the $\omega_{ns}^{(n)}$, we have

$$\omega_{kl}^{(n+1)} = \frac{1}{2}\left(\left(\delta_l^m \partial_k - \delta_k^m \partial_l\right) X^s - \left(\delta_l^s \partial_k - \delta_k^s \partial_l\right) X^m\right) \omega_{ms}^{(n)}.$$

This relation can be represented in the form

$$\omega_{kl}^{(n+1)} = \hat{J}_{kl}^{ms} \omega_{ms}^{(n)},$$

where the operator $\hat{J}$ is defined by equation (15). Therefore the invariant phase-space metric is defined by the following equation:

$$\omega_{kl}(\mathbf{x}, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\hat{J}^n)_{kl}^{ms} \omega_{ms}(\mathbf{x}, 0) = (\exp(t\hat{J}))_{kl}^{ms} \omega_{ms}(\mathbf{x}, 0).$$

As a result we have equation (16). □

Comments. We prove that equation (14) for the elements of phase-space metric can be expressed in terms of an operator $\hat{J}$ such that

$$\frac{\partial \omega_{kl}}{\partial t} = \hat{J}_{kl}^{ms} \omega_{ms}, \tag{17}$$

where $\hat{J}^{ab}_{kl}$ is defined by equation (15). In this case, the time evolution of the phase-space metric from initial condition $\omega_{kl}(\mathbf{x}_0, 0)$ to a value $\omega_{kl}(\mathbf{x}, t)$ at the time t can be written by the equation

$$\omega_{kl}(\mathbf{x}, t) = (\exp(t\hat{J}))^{ms}_{kl}\,\omega_{ms}(\mathbf{x}, 0). \tag{18}$$

Here the matrix exponential operator, $\exp(t\hat{J})$ can be called the metric propagator. The operator $\hat{J}$ can be considered as a metric analogue of the Liouville operator. Note that we can use the canonical coordinates (q, p) for $t = 0$ and the coordinate-independent initial metric: $\omega_{kl}(\mathbf{x}, 0) = \omega^{(0)}_{kl} = \text{const}$, $\partial_m \omega_{kl}(\mathbf{x}, 0) = 0$. Introducing time step $\Delta t = t/N$, we get

$$\omega_{kl}(\mathbf{x}, t) = (\exp(t\hat{J}))^{ms}_{kl}\,\omega_{ms}(\mathbf{x}, 0) = ([\exp(\Delta t\hat{J})]^N)^{ms}_{kl}\,\omega_{ms}(\mathbf{x}, 0).$$

It is natural to approximate the short-time propagator $\exp(t\hat{J})$ using the Trotter theorem [23–25]. The vector field $\mathbf{X} = \mathbf{X}(\mathbf{x})$ can be represented in the form

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2,$$

where $\mathbf{X}_1$ is a Hamiltonian term such that

$$X^k_1 = \omega^{kl}\frac{\partial H}{\partial x^l},$$

and $\mathbf{X}_2$ is a friction (non-Hamiltonian) term. As the result we have

$$(\hat{J}_{1;2})^{ms}_{kl} = \frac{1}{2}\left(\left(\delta^m_l \partial_k - \delta^m_k \partial_l\right)X^s_{1;2} - \left(\delta^s_l \partial_k - \delta^s_k \partial_l\right)X^m_{1;2}\right). \tag{19}$$

An important consequence would be the ability to formulate rigorous numerical integration algorithms based on Trotter-type splittings [23–25] of the classical propagator $\exp(\Delta t\hat{J})$. The metric propagator for the small time step is

$$\exp[\Delta t(\hat{J}_1 + \hat{J}_2)] = \exp\left[\tfrac{1}{2}\Delta t\hat{J}_2\right]\exp[\Delta t\hat{J}_1]\exp\left[\tfrac{1}{2}\Delta t\hat{J}_2\right] + O(\Delta t^3).$$

Finally, we obtain

$$\exp[t\hat{J}] = \exp[t(\hat{J}_1 + \hat{J}_2)] = \left(\exp\left[\tfrac{1}{2}\Delta t\hat{J}_2\right]\exp[\Delta t\hat{J}_1]\exp\left[\tfrac{1}{2}\Delta t\hat{J}_2\right]\right)^N + O(\Delta t^3).$$

5. Poisson brackets for non-Hamiltonian systems

Let us consider the skew-symmetric tensor field $\omega^{kl} = \omega^{kl}(\mathbf{x}, t)$ that is defined by the equations

$$\omega^{kl}(\mathbf{x}, t)\omega_{lm}(\mathbf{x}, t) = \omega^{lk}(\mathbf{x}, t)\omega_{ml}(\mathbf{x}, t) = \delta^k_l.$$

As the result this tensor field satisfies the Jacobi identity

$$\omega^{kl}\partial_l\omega^{ms} + \omega^{ml}\partial_l\omega^{sk} + \omega^{sl}\partial_l\omega^{km} = 0.$$

It follows from the Jacobi identity for ω_{kl}.

In proposition 3, we suggest the time-dependent phase-space metric $\omega_{kl}(\mathbf{x}, t)$, which satisfies the Jacobi identity. As the result we have Lie algebra that is defined by the following brackets:

$$\{A, B\} = \omega^{kl}(\mathbf{x}, t)\partial_k A\partial_l B. \tag{20}$$

It is easy to prove that these brackets are Poisson brackets.

In the general case, the rule of term-by-term differentiation with respect to time that has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\{A, B\} = \{\dot{A}, B\} + \{A, \dot{B}\} \tag{21}$$

where $\dot{A} = \mathrm{d}A/\mathrm{d}t$ is not valid for non-Hamiltonian systems. In the general case, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\{A, B\} = \{\dot{A}, B\} + \{A, \dot{B}\} + J(A, B)$$

where

$$J(A, B) = \omega^{kl}_{(1)}(\mathbf{x}, t)\partial_k A \partial_l B, \qquad \omega^{kl}_{(1)}(\mathbf{x}, t) = \omega^{km}\omega^{ls}(\partial_s X_m - \partial_m X_s).$$

Note that time evolution of the Poisson brackets (20) for non-Hamiltonian systems can be considered as t-deformation [28] of the Lie algebra in phase-space.

If we use the invariant phase-space metric, then rule (21) is valid. As the result the suggested Poisson brackets (20) of two constants of motion is a constant of motion and rule (21) is satisfied.

6. Example: system with linear friction

Let us consider the non-Hamiltonian system that is described by the equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p^i}, \qquad \frac{\mathrm{d}p^i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} - K^i_j(t)p^j, \tag{22}$$

where $H = T(p) + U(q)$ and $i; j = 1, \ldots, n$. Here $T(p)$ is a kinetic energy, $U(q)$ is a potential energy. The term $-K^i_j(t)p^j$ is a friction term. Usually, this system is described by the phase-space metric ω_{kl} that has the form

$$\|\omega_{kl}\| = \begin{pmatrix} 0 & G \\ -G^{\mathrm{T}} & 0 \end{pmatrix}, \tag{23}$$

where the matrix $G = \|g_{ij}\|$ is equal to identity matrix $E = \|\delta_{ij}\|$. Here G^{T} is the transpose matrix for the matrix G. The symplectic form is defined by equation (4) in the form

$$\omega = 2g_{ij}\,\mathrm{d}q^i \wedge \mathrm{d}p^j = \delta_{ij}\,\mathrm{d}q^i \wedge \mathrm{d}p^j,$$

where $g_{ij} = (1/2)\delta_{ij}$. The phase-space compressibility

$$\kappa = \sum_{i=1}^{n}\left(\frac{\partial \dot{q}^i}{\partial q^i} + \frac{\partial \dot{p}^i}{\partial p^i}\right) \tag{24}$$

of system (22) is defined by the spur of the matrix $K = \|K^i_j\|$ in the form

$$\kappa = -Sp\|K^i_j\|.$$

Proposition 5. *The invariant phase-space metric for the classical system (22) has the form*

$$\|\omega_{kl}(t)\| = \begin{pmatrix} 0 & G(t) \\ -G^{\mathrm{T}}(t) & 0 \end{pmatrix}, \tag{25}$$

where the matrix G is defined by

$$G(t) = G(t_0)\exp\int_{t_0}^{t}\left\|K^i_j(\tau)\right\|\,\mathrm{d}\tau. \tag{26}$$

Proof. Suppose that phase-space metric depends on time t. Therefore the matrix G and elements g_{kl} are the functions of the variable t: $G = G(t)$, $g_{ij} = g_{ij}(t)$. Let us consider the total time derivative of the symplectic form

$$\omega = 2g_{ij}(t)\,\mathrm{d}q^i \wedge \mathrm{d}p^j.$$

As the result, we have

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = 2\left(\frac{\mathrm{d}g_{ij}}{\mathrm{d}t} - g_{im}K^m_j\right)\mathrm{d}q^i \wedge \mathrm{d}p^j.$$

In order to have the invariant phase-space metric ($\mathrm{d}\omega/\mathrm{d}t = 0$), we use the following equation:

$$\frac{\mathrm{d}g_{ij}}{\mathrm{d}t} = g_{im} K^m_j .$$

The matrix $G = \|g_{ij}\|$ is satisfied by the matrix equation

$$\frac{\mathrm{d}G(t)}{\mathrm{d}t} = G(t) K(t), \tag{27}$$

where $K(t) = \|K^i_j(t)\|$ is a matrix of friction coefficients. Suppose that $G(t_0) = E$, where E is the identity matrix. The solution of equation (27) has the form

$$G(t) = G(t_0) \exp \int_{t_0}^{t} K(\tau)\, \mathrm{d}\tau = \exp \int_{t_0}^{t} K(\tau)\, \mathrm{d}\tau. \tag{28}$$

If the matrix K is diagonal matrix with elements $K^i_j(t) = K_j(t)\delta^i_j$, then we have the matrix elements

$$g_{ij}(t) = \delta_{ij} \exp \int_{t_0}^{t} K_j(\tau)\, \mathrm{d}\tau.$$

As the result, the invariant phase-space metric for system (22) is defined by equations (25) and (26). □

The metric determinant factor [4–6]

$$\sqrt{g(\mathbf{x}, t)} = \sqrt{\det\|\omega_{kl}(\mathbf{x}, t)\|}$$

for the phase-space metric (23) is defined by the relation $\sqrt{g} = \sqrt{\det(GG^{\mathrm{T}})}$. Using $\det AB = \det A \det B$ and $\det A^{\mathrm{T}} = \det A$, we have the metric determinant factor in the form $\sqrt{g} = |\det G|$. If the matrix G has the form (28), then

$$\sqrt{g(t)} = \left| \det \exp \int_{t_0}^{t} K(\tau)\, \mathrm{d}\tau \right| .$$

Using $\det \exp A = \exp \mathrm{Sp}\, A$, we get that the invariant phase-space metric has the determinant that is connected with the phase-space compressibility [4–6] by the equation

$$\sqrt{g(t)} = \exp \int_{t_0}^{t} \mathrm{Sp}\, K(\tau)\, \mathrm{d}\tau = \exp - \int_{t_0}^{t} \kappa(\tau)\, \mathrm{d}\tau, \tag{29}$$

where κ is the phase-space compressibility (24).

For example, the system

$$\frac{\mathrm{d}q_1}{\mathrm{d}t} = \frac{p_1}{m}, \qquad \frac{\mathrm{d}p_1}{\mathrm{d}t} = -\frac{\partial U(q)}{\partial q_1} - K_1 p_1,$$
$$\frac{\mathrm{d}q_2}{\mathrm{d}t} = \frac{p_2}{m}, \qquad \frac{\mathrm{d}p_2}{\mathrm{d}t} = -\frac{\partial U(q)}{\partial q_2} - K_2 p_2,$$

has the invariant phase-space metric $\omega_{kl}(t)$ in the form

$$\|\omega_{kl}(t)\| = \begin{pmatrix} 0 & 0 & \mathrm{e}^{K_1 t} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{K_2 t} \\ -\mathrm{e}^{K_1 t} & 0 & 0 & 0 \\ 0 & -\mathrm{e}^{K_2 t} & 0 & 0 \end{pmatrix}. \tag{30}$$

In this case, the metric determinant factor is equal to

$$\sqrt{g(t)} = \sqrt{\det\|\omega_{kl}\|} = \mathrm{e}^{(K_1+K_2)t}.$$

7. Conclusion

Tuckerman *et al* [4–6] suggest a formulation of non-Hamiltonian statistical mechanics which uses the invariant phase-space measure. The invariant measure is connected with phase-space metric. Tuckerman *et al* consider the properties of only the determinant $g(\mathbf{x}, t)$ of the metric. The phase-space metric is not considered in [4–6]. We consider the invariant phase-space metric for non-Hamiltonian systems. The proposal to use an invariant time-dependent metric determinant factor in the volume element corresponds precisely to finding a skew-symmetric phase-space metric (symplectic form) that is an integral of motion. Therefore we consider the skew-symmetric metric.

Sergi [10, 11] uses the skew-symmetric phase-space metric that is not satisfied by the Jacobi identity. As the result the generalization of the Poisson brackets for non-Hamiltonian systems leads one to non-Lie algebra. Note that non-Lie algebra for non-Hamiltonian systems is considered in [29]. In this paper we consider an invariant antisymmetric phase-space metric that satisfies the Jacobi identity, and defines the Lie algebra in phase-space. We call the metric is invariant if the metric tensor field is an integral of motion ($\mathrm{d}\omega/\mathrm{d}t = 0$). This invariant phase-space metric $\omega_{kl}(\mathbf{x}, t)$ defines the invariant phase-space measure v by the equation $v = (1/n!)\omega^n = \sqrt{g(\mathbf{x}, t)}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{2n}$, where $g(\mathbf{x}, t)$ is the metric determinant $g(\mathbf{x}, t) = \det(\omega_{kl}(\mathbf{x}, t))$ and $\mathrm{d}v/\mathrm{d}t = 0$. The suggested time-dependent skew-symmetric phase-space metric leads to a constant value of the entropy density, so that the associated distribution function obeys an evolution equation associated with incompressible dynamical flow.

Note that the invariant phase-space metric of some non-Hamiltonian systems can lead us to the lack of smoothness of the metric. In this case, the phase-space probability distribution can be collapsed onto a fractal set of dimensionality lower than in the Hamiltonian case [26, 27]. Unfortunately the description of lack of smoothness in [26, 27] is considered without using the curved phase-space approach [4–6]. Note that classical systems that are Hamiltonian systems in the usual phase-space are non-Hamiltonian systems in the fractional phase-space [13, 32].

In the papers [30, 31], the quantization of the evolution equations for non-Hamiltonian and dissipative systems was suggested. Using this quantization it is easy to derive the quantum analogue of the invariant Poisson brackets, which satisfy the rule of term-by-term differentiation with respect to time.

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